Research Statement

Xiang Wan

My main research interests are nonlinear Ordinary/Partial Differential Equations (ODEs/PDEs), on both the pure and applied sides. In particular, my research includes three main components:

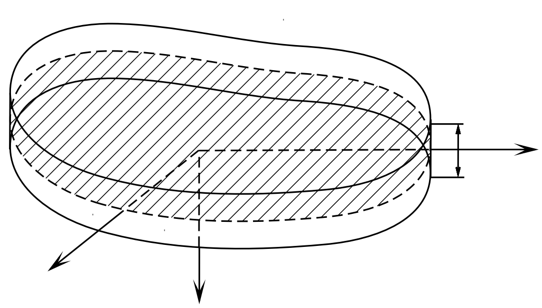
- Studying the modeling of natural phenomena, for instance physical kinematics or biological dynamics, via PDE systems.
- Understanding these systems theoretically by mathematical tools mainly by Analysis, including classical and modern PDE Theory, Functional Analysis, Semigroup Theory, Mathematical Control Theory, etc.
- Developing Numerical Methods to simulate these systems with a two-fold aim: (i) to give impetus to, and stimulate, intriguing theoretical questions; (ii) to verify and illustrate properties and characteristics that have been already mathematically proved.

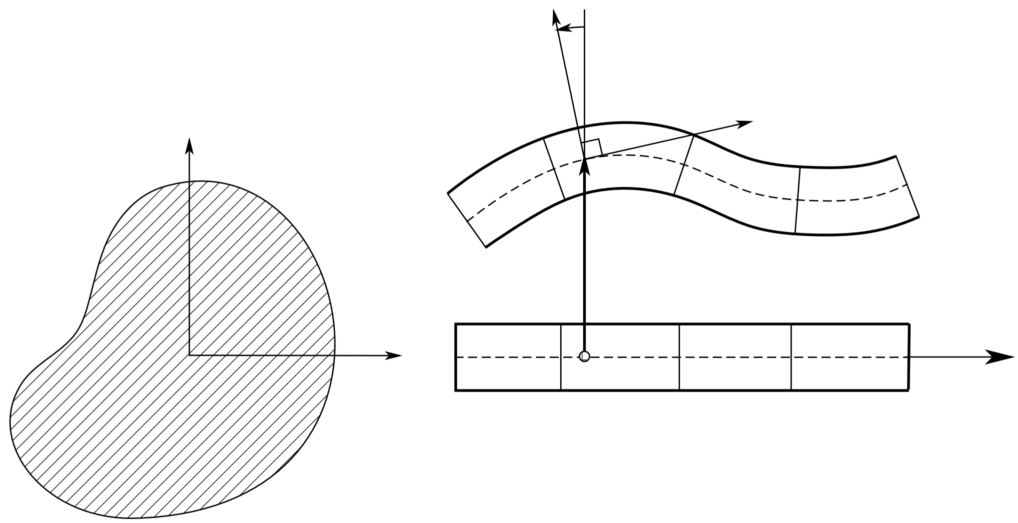
In order to achieve these goals, I have selected and pursued collaborative and interdisciplinary researches with my thesis advisor, Prof. Irena Lasiecka, my colleagues from different institutions, and an undergraduate student at the University of Virginia.

Project 1: Nonlinear Thermo-elastic Kirchhoff–Love Plate System

The goal of this project is to first establish a theoretic framework, including global well-posedness and long time behavior, for the nonlinear thermo-elastic Kirchhoff–Love plate equations, and secondly to understand how properties of the plate (thickness, geometry, magneto-elasticity, etc.) effect the performance of the system.

1.1 Modeling In continuum mechanics, plate theories mathematically describe the mechanics of flat plates which are defined as plane structural elements with a small and uniform thickness compared to the planar dimensions. Of the numerous plate theories that were developed since the late 19^{th} century, one of the most famous and successful attempts is the Kirchhoff–Love theory of plates, an extension of Euler-Bernoulli beam theory, which was developed in 1888 by Love[Lo.1] using assumptions proposed by Kirchhoff. The theory asserts that a mid-surface plane can be used to represent a three-dimensional plate in a two-dimensional form, together with some kinematic assumptions that are made in this theory.





In addition to mechanical loads on Kirchhoff–Love plates, J. Lagnese and J. L. Lions in 1988 considered the contribution to deformation both by the elasticity of the material and by the temperature distribution over the plate, while assuming that the plate is elastically and thermally isotropic[Lag-Lions.1].

1.2 Background Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary representing the mid-plane of a thermoelastic plate. With w and θ denoting the vertical deflection and an appropriately weighted thermal moment with respect to the thickness, both depending on a scaled time variable $t > 0$ and the space variable $(x_1, x_2) \in \Omega$, the nonlinear Kirchhoff–Love thermo-elastic plate system reads as:

$$w_{tt} - \gamma \triangle w_{tt} + a(-\triangle w)\triangle^2 w + \alpha \triangle \theta = f(-\triangle w, -\nabla \triangle w) \quad \text{in } (0,\infty) \times \Omega, \tag{1a}$$

$$\beta \theta_t - \eta \triangle \theta + \sigma \theta - \alpha \triangle w_t = 0 \quad \text{in } (0,\infty) \times \Omega \tag{1b}$$

together with the "hinged"/Dirichlet boundary conditions

$$w = \triangle w = \theta = 0 \text{ in } (0,\infty) \times \Omega \tag{1c}$$

and the initial conditions

$$w(0,\cdot) = w^0, \quad w_t(0,\cdot) = w^1, \quad \theta(0,\cdot) = \theta^0 \text{ in } \Omega. \tag{1d}$$

Justification of the nonlinearity in Equation (1a) is given in [La-Po-Wan.1]. Here, $\alpha, \beta, \gamma, \eta, \sigma$ are positive constants and $a\colon \mathbb{R} \to (0,\infty)$ as well as $f\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are smooth functions. For thin plates, γ behaves like h^2 as $h \to 0$, where h denotes the (uniform) thickness of the plate. The limiting case $\gamma = 0$ corresponds to the von Kármán thermoelastic plate model.

1.3 Prior Relevant Work There were many studies on simplifications/variations of this model. However, the differences turn out to be critical. This makes our work a breakthrough in two ways: i) no similar result was achieved before for nonlinear thermo-elastic Kirchhoff–Love plate equations; ii) the techniques used in previous studies **fail** to work in our case. In particular,

1. Linear Euler-Bernoulli thermo-elastic plates: $\gamma = 0, f \equiv 0$. Lasiecka, Kim, Renardy, Triggiani, et. al. showed the analyticity in $L^2(\Omega)$ of the s.c. contraction semigroup generated by the linear equations with respect to all five canonical different sets of B.C.[Kim.1, Lag.1, La-Tr.3, Liu-Ren.1], including coupled B.C.. More recently, a L_p theory of the analyticity of the semigroup was shown by Rache et. al.[De-Ra-Sh.1]. From this, the well-posedness and uniform stability of the system follow.

2. Nonlinear Euler-Bernoulli thermo-elastic plates: $\gamma = 0$ with specific form of $f = \Delta((\Delta w)^3)$, originating from the magneto-elasticity of the material of the plate. Lasiecka, Maad, and Sasane studied extensively the weak solution in $L^2(\Omega)$ of the quasilinear model with zero B.C[La-Ma-Sa.1]. In this work, a well-posedness result for "small" initial data was established via the Maximal Parabolic Regularity[De-Hi-Pr.1, Pr.1, Pr-Sim.1]. A later work by Lasiecka and Wilke[La-Wi.1] extended this result to a L_p-framework for $p \in (n/2 + 2, \infty)$: here the initial condition still need to be small, but can be taken less regular.

1.4 Differences with our model (1a–d) In contrast, however, our model (1.1a–d) is **no longer** a quasilinear parabolic system, but instead a hyperbolic–parabolic system. Indeed, Lasiecka and Triggiani pointed out the lack of analyticity of the linear semigroup[La-Tr.4]. A second consequence is that the Maximal Parabolic Regularity does not apply for our model. Therefore, we had to change stretegy. We turned into the energy method to show the following results.

1.3 Local Well-posedness In our work we studied *classical* solutions. The central strategy is to find the correct energy space, namely, appropriate multipliers, so that the energy estimate would work.

Definition 1. *Under a classical solution at the energy level $s \geq 2$ to Equations (1a–d) on the time interval $[0, T]$ we understand a function pair $(w, \theta)\colon [0, T] \times \bar{\Omega} \to \mathbb{R} \times \mathbb{R}$ such that it satisfies Equations (1a–d), and possesses the following regularity:*

$$z \in \Big(\bigcap_{m=0}^{s-1} C^m\big([0, T], H^{s-m}(\Omega) \cap H_0^1(\Omega)\big)\Big) \cap C^s\big([0, T], L^2(\Omega)\big), \tag{2}$$

$$\theta \in \Big(\bigcap_{m=0}^{s-2} C^m\big([0, T], H^{s-m}(\Omega) \cap H_0^1(\Omega)\big)\Big) \cap C^{s-1}\big([0, T], L^2(\Omega)\big) \cap H^{s-1}\big(0, T; H^1(\Omega)\big) \tag{3}$$

where we have set

$$z := -\Delta w \quad in\ \Omega, \quad w\big|_\Gamma = 0; \qquad or \qquad z = (A)_D w; \qquad or \qquad w = (A)_D^{-1} z. \tag{4}$$

A classical solution on the interval $[0, \infty)$ is defined correspondingly.

Assumption 2. *[La-Po-Wan.1, Assumption 4] Let $s \geq 3$ be an integer. We assume that the functions a and f satisfy some differentiability conditions; that the initial data satisfy certain regularity conditions; and furthermore some compatibility conditions on lower energy levels.*

The integer s corresponds to the energy level at which we establish our theory. Works on linear and nonlinear Euler-Bernoulli systems hold at the level $s = 1$ (with a slight and compatible variation of definition). The theoretic obstacles mentioned in our previous discussion specifically make our work *not* true on the low energy levels, specifically for $s = 1$ or 2. Consequently, we had to build up a three-level energy estimate in order to comply with the compatibility condition proposed in Assumption 2.

Theorem 3. *For $s = 3$, there exists a time period $T > 0$ such that (1a–d) possess a unique classical solution (w, θ) at the energy level $s = 3$.*

We constructed two different proofs. One is based on the Banach Fixed Point Theorem. By investigating the linearized system, we constructed a solution map Φ that is a contraction mapping a small energy ball to itself. The other proof is based on a modified version of the existence theories from [Ra-Ji.1].

1.4 Uniform Stability and Global Well-posedness Theorem 3 provides the existence and uniqueness in a (possibly short) finite time interval, yet another important question we want to answer is the long time behavior of the system. One would hope that local solution does not "blow up" in the inherited energy space, and maybe even decay at certain rate so that the system would stabilize. Although this is often the case intuitively by the Law of Conservation of Energy, mathematically it takes a lot of effort to show. Our proof relies on the following key lemma:

Lemma 4. *[La-Po-Wan.1, Lemma 6] Let Assumption 2 be satisfied, and let T be the positive time period from Theorem 3 and $X_3(t)$ be energy at the level $s =$, then*

$$X_3(T) + \int_0^T X_3(\tau) \le C_1 X_3(0) + C_2 \sum_{i \in I} X_3(T)^{\alpha_i} + C_3 \sum_{j \in J} \int_0^T X_3(\tau)^{\beta_j} \tag{5}$$

where $C_k \ge 0$, $k = 1, 2, 3$ are constants; both I and J are some finite subsets of $\mathbb{N}$; $\alpha_i > 1$ for any $i \in I$, and $\beta_j > 1$ for any $j \in J$.

A critical fact is that α_i and β_j are strictly greater than 1. Under the assumption that $X(0)$ is small enough, we further argue that the finite sums on the RHS are uniformly bounded by the two terms on the LHS, and therefore the uniform stability of energy E_s and global well-posedness of the solution follow:

Theorem 5. *[La-Po-Wan.1, Theorem 7 & 8] The solution from Theorem 3 is global, namely, T could be taken to be ∞. Moreover, there exists a positive number ρ such that if $E_s(0) \le \rho$, then the energy of the system at level 3 decays exponentially.*

Remark 6. *Throughout the proof, the coupling terms $\alpha\Delta\theta$ in (1a) and $-\alpha\Delta w_t$ in (1b) plays a fundamental role in two ways: i) the decay of energy is steered from the thermal diffusion of heat transfer (parabolic equation) to elastic deformation (hyperbolic equation) through the coupling; ii) w is naturally one level lower in space regularity, but the regularities of w_t and θ support a boost of w-regularity through the coupling.*

Project 2: Schrödinger's Equation with Finite Rank Boundary Feedback Control

This project arose during a seminar that I followed during one of my several visits to the Department of Mathematical Sciences, University of Memphis, where my PhD advisor Prof. Irena Lasiecka and my former professor of many graduate courses at UVa, Prof. Roberto Triggiani, have moved to two years ago.

It suffices to consider the problem in its simplest form, leaving to subsequent research efforts the extension to the nonlinear case. Thus, consider at first the following mixed problem for the linear Schrödinger's equation defined on a bounded multi-dimensional domain Ω, with sufficiently smooth boundary Γ.

$$\begin{aligned} iy_t + \Delta y &= 0 && \text{in } (0,T] \times \Omega = Q, && \text{(6a)} \\ y\big|_{t=0} &= y_0 && \text{in } \Omega, && \text{(6b)} \\ y\big|_{\Sigma} &= (y,w)g && \text{in } (0,T] \times \Gamma = \Sigma, && \text{(6c)} \end{aligned}$$

where w and g are two arbitrary elements of say $L_2(\Omega)$ and $L_2(\Gamma)$, respectively. The problem is non-standard as one requires the Dirichlet boundary term to be expressed in feedback form as a, say, rank-one term as in (6c).

The question is to establish the well-posedness in the optimal space of regularity of the above closed-loop feedback Schrödinger's equation with finite rank feedback control. We have a strategy for solving the above problem. To this end, one seeks to use a surprising but optimal regularity result[La-Tr.1] of the Schrödinger's equation with open-loop Dirichlet bondary control u

$$\begin{aligned} & iv_t + \Delta v = 0 \;\text{ in } Q; \quad v_0 = 0 \;\text{ in } \Omega; \quad v\big|_{\Sigma} = u \in L_2(0,T; L_2(\Gamma)) \\ \Longrightarrow\; & v \in C([0,T]; H^{-1}(\Omega)), \quad \text{continuously.} \end{aligned} \tag{7}$$

We emphasize that the space $H^{-1}(\Omega)$ is optimal for this problem. By duality, the above optimal regularity result: Dirichlet boundary control $\rightarrow$ interior regularity can be expressed equivalently as optimal trace regularity from the Initial datum to the normal derivative on the boundary of the Schrödinger's problem this time with zero Dirichlet B.C.

$$\begin{aligned} & i\phi_t + \Delta\phi = 0 \;\text{ in } Q; \quad \phi_0 \in H_0^1(\Omega), \quad \phi\big|_{\Sigma} = 0 \\ \Longrightarrow\; & \frac{\partial\phi}{\partial\nu}\bigg|_{\Sigma} \in L^2(0,T; L^2(\Gamma)), \quad \text{continuously.} \end{aligned} \tag{8}$$

These optimal results were established by Lasiecka-Triggiani in 1992[La-Tr.2].

To solve our feedback problem we shall use as guide the corresponding closed-loop feedback problem for second order hyperbolic equations with finite rank boundary feedback control (in either the Dirichlet or in the Neumann B.C.) which was solved in the mid-80s by Lasiecka-Triggiani[La-Tr.1]. The above feedback well-posed problem can also be posed in a Riemannian setting, over a Riemann manifold with boundary. These problems are very relevant in boundary control theory for Partial Differential Equations; for instance in the framework of boundary stabilization (see future project below).

Project 3: Qualitative Analysis and Optimal Control of Cancer Modeling

This is a survey work from my Master Thesis[Wan.1] at the Department of Mathematics and Statistics, Auburn University, Alabama. We studied four systems of ordinary differential equations, which model the interrelationships between different cell populations in the presence of tumor cells as well as applications of treatments, such as immunotherapy, chemotherapy, and radiotherapy.

For the first two models, we only consider the case with radiation treatment. One model takes in account a single general cell population with its corresponding radiated cell population; the other model looks at the host and tumor cell populations together with their corresponding radiated cell populations. Two different kinds of radiation are studies separately: constant radiation and decaying radiation.

For the third and fourth models, we consider both the immunotherapy and chemotherapy. The third model includes three cell populations: host cells, tumor cells, and immune cells, as well as the drug concentration. In the fourth model we studied the properties of its null-surfaces, equilibria and the corresponding stabilities. Not only did we extended the previous model into one of six populations, with the immune cells in the third model being specified into three different ones: $CD8^{+}T$ cells, circulating lymphocytes, and IL-2. but also we focused on the situation when controls are added in a linear manner. We investigated the existence of optimal controls and find the characterization of optimal bang-bang controls.

On-going and Future Research Projects

1. Theoretic study of Thermo-elastic Plates Now with the fundamental framework on the system (1a–b) established, some natural yet interesting extensions of our work arise:

- Different sets of B.C., physically more relevant and mathematically more challenging. They include clamped B.C. but also coupled boundary conditions to include the physically most relevant and mathematically most challenging case of so called "free" B.C. (they include 2^{th} and 3^{rd} mechanical boundary operators as well as coupled Robin thermal B.C. All of these have been studied with linear Euler-Bernoulli plates, but no work has been done with the nonlinear Kerchhoff-Love plates.

- A different law of heat transfer. Instead of using the Fourier equation to model the heat conduction, I would like to consider instead the Relativistic heat conduction (Cattaneo-Vernotte Model). By employing the finite speed of propagation of the heat transfer, we are changing the current hyperbolic-parabolic system into a completely hyperbolic system. However, due to the presence of the coupling as analyzed after Theorem 5, there is a big chance that we will be able to achieve similar result at some energy level.

2. Numerical Simulation of Thermo-elastic Plates This is an on-going project. A preliminary investigation via Matlab® illustrated some of our stability results. A further work will lie on different sets of boundary conditions as well as more complicated geometries of domains. For example, properties on the same systems with free mechanical and Robin thermal B.C. is completely unknown, and it will be really interesting to see the numerical behavior before starting any detailed theoretic study.

This project is due to be submitted for the "Double Hoo Research Grant" at the University of Virginia (paired with an undergraduate) in February 2016.

3. Stabilization of The linear and nonlinear Schrödinger's Equation by means of finite rank boundary controls

- **Stabilization of linear case.** With reference to the feedback system (6a–c), if, say, $g \equiv 0$, then we have the case of a unitary group: $\|y(t)\| \equiv \|y_0\|$. Next, can we select the vectors $\{w, g\}$ such that the corresponding problem is **strongly stable** (in the topology of the optimal regularity space Y of the well-posedness)?

$$\|y(t, y_0)\| \to 0 \quad \text{as } t \to +\infty, \quad \forall\, y_0 \in Y. \tag{9}$$

  Furthermore, assume now $y_0 \in \mathcal{D}(A)$, A being the generator of the feedback semigroup describing the well-posedness of the feedback problem. Then we seek **polynomial** or **rational** stability, that is we seek to select the vectors $\{w, g\}$ such that

$$\|y(t, y_0)\| \le \frac{C}{t^{\alpha}} \|y_0\|_{\mathcal{D}(A)} \tag{10}$$

for the optimal α. For this problem, we shall use the strategy of the recent paper[Av-La-Tr.1] in fluid-structure interaction which is based on two main approaches:
(i) the use of the recent result by Borichev-Tomilov[Bo-To.1], which provides a characterization of the polynomial/algebraic decay α by means of an equivalent condition of the resolvent operator $R(is, A)$ on the imaginary axis ('frequency domain approach');
(ii) the use of micro-local analysis as in [Av-La-Tr.1] to obtain the optimal α.

- **Stabilization of nonlinear case** by the same boundary feedback. Here our guide will be papers [La-Lu.1] and [Lu.1] on the nonlinear case of the fluid-structure interaction model.

References

[Av-La-Tr.1] G. Avalos, I. Lasiecka, and R. Triggiani, A PDE Heat–Structure Interaction: Optimal Rational Decay Rate by Micro-local Analysis, 2015.

[Bo-To.1] A. Borichev and Y. Tomilov, Optimal polynomial decay of functions and operator semigroups, *Math. Ann.* 347(2) (2010), 455–478.

[De-Hi-Pr.1] R. Denk, M. Hieber, and J. Prüss, $\mathcal{R}$-boundedness, Fourier Multipliers and Problems of elliptic and parabolic type, *Memoirs of the AMS*, no. 788, 2003.

[De-Ra-Sh.1] R. Denk, R. Racke, and Y. Shibata, L_p theory for the linear thermoelastic plate equations in bounded and exterior domains, *Advances in Differential Equations* 14.7/8 (2009): 685-715.

[Kim.1] J. U. Kim, On the energy decay of a linear thermo-elastic bar and plate, *SIAM J. Math. Anal.* 23 (1992), 889–899.

[Lag.1] J. Lagnese, *Boundary Stabilization of Thin Plates*, SIAM, Philadelphia (1989).

[Lag-Lions.1] J. Lagnese and J.-L. Lions, *Modelling analysis and control of thin plates*, Recherches en Mathématiques Appliqués (1988).

[La-Lu.1] I.Lasiecka, Y. Lu, Interface feedback control stabilization of a nonlinear fluid-structure interaction. Nonlinear Anal. 75 (2012), no. 3, 1449–1460.

[La-Po-Wan.1] I. Lasiecka, M. Pokojovy, and X. Wan, Global Existence and Stability for a Nonlinear Thermoelastic Kirchhoff-Love Plate, submitted.

[La-Ma-Sa.1] I. Lasiecka, S. Maad, and A. Sasane, Existence and exponential decay of solutions to a quasilinear thermoelastic plate system, *Nonlinear Differential Equations and Applications NoDEA* 15.6 (2008): 689-715.

[La-Tr.1] I. Lasiecka and R. Triggiani, Finite rank, relatively bounded perturbations of semigroups generators, *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze* 12(4) 1985, 641–668.

[La-Tr.2] I. Lasiecka and R. Triggiani, Optimal regularity, exact controllability and uniform stabilization of the Schrödinger equation, *Diff. Int. Eqns.* 5 (1992), 521–535.

[La-Tr.3] I. Lasiecka and R. Triggiani, Two direct proofs of analyticity of the s.c. semigroup arising in abstract thermo-elastic equations, *Advances in Differential Equations* 3.3 (1998): 387-416.

[La-Tr.4] I. Lasiecka and R. Triggiani, Analyticity, and lack thereof, of thermo-elastic semigroups, *ESAIM: Proceedings.* Vol. 4. EDP Sciences, 1998.

[La-Wi.1] I. Lasiecka and M. Wilke, Maximal regularity and global existence of solutions to a quasilinear thermoelastic plate system, *Dis. & Cont. Dyn. Sys.*, vol 33, Nr 11/12, pp 5189-5202, 2013.

[Liu-Ren.1] Z. Liu. and M. Renardy, A note on the equations of a thermoelastic plate, *Appl. Math. Lett.* 8 (1995), No. 3, pp. 1-6.

[Lo.1] A. E. H. Love, On the small free vibrations and deformations of elastic shells, *Philosophical Transactions of the Royal Society of London. A* (1888): 491-546.

[Lu.1] Lu, Yongjin, Uniform stabilization to equilibrium of a nonlinear fluid-structure interaction model, *Nonlinear Anal. Real World Appl.* 25 (2015), 51–63.

[Pr.1] J. Prüss, Maximal regularity for evolution equations in L_p spaces. *Conf. Semin. Mat. Univ. Bari*, no. 285 (2002), 1–39.

[Pr-Sim.1] J. Prüss and G. Simonett, Maximal regularity for evolution equations in weighted L_p spaces, *Arch. Math.*, 82, no. 5 (2004), 415–431.

[Ra-Ji.1] R. Racke and S. Jiang, *Evolution Equations in Thermoelasticity*, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2000.

[Wan.1] X. Wan, Some Systems of Ordinary Differential Equations from Cancer Modeling: Qualitative Analysis and Optimal Treatment, Master Thesis (2011), Auburn University, Alabama.